ISRAEL JOURNAL OF MATHEMATICS, Vol. 22, No. 2, 1975

$\mathscr{L}_1$ SUBSPACES OF l_1

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ABSTRACT

It is proved that every subspace of l_1 which is an $\mathscr{L}_{1,\lambda}$ space with λ close enough to 1 is isomorphic to l_1.

A Banach space X is called a $\mathscr{L}_{1,\lambda}$ *space* $(\lambda > 1)$ if for every finite dimensional subspace E of X there is a finite dimensional subspace F of X such that $E \subset F$ and $d(F, l_1^n) \leqq \lambda$ (here $n = \dim F$ and $d(A,B)$, the Banach-Mazur distance between the isomorphic spaces A and B, is defined by $d(A,B) = \inf\colon \{\|T\|\,\|T^{-1}\|\colon T$ an invertible operator from A onto $B\})$. The space X is said to be a $\mathscr{L}_1$ *space* if it is a $\mathscr{L}_{1,\lambda}$ space for some $\lambda > 1$. For the basic properties of $\mathscr{L}_1$ spaces and their important role in the study of the classical spaces the reader is referred to [5] and [6]. It is easy to see that every $L_1(\mu)$ space is a $\mathscr{L}_{1,\lambda}$ space for every $\lambda > 1$ but the family of $\mathscr{L}_1$ spaces is much richer than the family of $L_1(\mu)$ spaces (see [4]). Even the space l_1, in spite of being a very "small" $\mathscr{L}_1$ space, contains $\mathscr{L}_1$ subspaces which are not isomorphic to l_1. Indeed, as is shown in [3], if $\{e_n\}$ denotes the unit vector basis of l_1 and $x_n = e_n - \frac{1}{2}(e_{2n+1} + e_{2n+2})$, then $X = \operatorname{span}\{x_n\}$ is a $\mathscr{L}_{1,2+\varepsilon}$ space for each $\varepsilon > 0$ but X is not isomorphic to l_1.

The purpose of this paper is to show that every $\mathscr{L}_{1,\lambda}$ subspace of l_1 is isomorphic to l_1 if λ is close enough to 1. More precisely,

THEOREM. *Let X be a subspace of l_1 and assume that X is a $\mathscr{L}_{1,\lambda}$ space with $\lambda \leqq 1.02$. Then X is isomorphic to l_1.*

Our main tool is the following recent result of L. E. Dor,

PROPOSITION 1 (cf. [1] Theorem A). *Let E be a finite dimensional subspace of l_1 and assume that $d(E, l_1^n) \leqq \lambda$ where $\lambda < 2^{\frac{1}{2}}$. Then there is a projection P of l_1 onto E with $\|P\| \leqq \lambda[1 - (2 + 2^{\frac{1}{2}})(1 - \lambda^{-1})]^{-1}$.*

† This research has been supported in part by the N.S.F. Grant GP 33578.
Received February 14, 1975

We also need

PROPOSITION 2 (cf. Proposition 1 of [2]). *Let* $1 \leqq \lambda$, $\theta < 2$ *and let* $\{x_j\}_{j=1}^n$ *be elements of* l_1 *satisfying*

$$\lambda^{-1}\sum_1^n |a_j| \leqq \left\| \sum_1^n a_j x_j \right\| \leqq \sum_1^n |a_j|$$

for all sequences $\{a_j\}_{j=1}^n$ *of scalars. If there is a projection* P *of* l_1 *onto* $\operatorname{span}\{x_j\}_{j=1}^n$ *with* $\|P\| \cdot \lambda \leqq \theta$, *then there are mutually disjoint sets* $\{A_j\}_{j=1}^n$ *of integers such that for each* j, $\|\Sigma_{i \in A_j} x_j(i) e_i\| \geqq 2\theta^{-1} - 1$, *where* $\{e_i\}_1^\infty$ *denotes the unit vector basis of* l_1.

Combining Proposition 1 and Proposition 2 we get the following

PROPOSITION 3. *Let* E *be a finite dimensional subspace of* l_1, *let* $\lambda > 1$ *satisfy the inequality* $\lambda^2[1-(2+2^{\frac{1}{2}})(1-\lambda^{-1})]^{-1} < 2$ *and let* $\{x_j\}_{j=1}^n$ *be a basis of* E *satisfying*

$$\lambda^{-1}\sum_{j=1}^n |a_j| \leqq \left\| \sum_1^n a_j x_j \right\| \leqq \sum_1^n |a_j|$$

for all sequences $a_1, a_2, \cdots, a_n$ *of scalars. Then there exist mutually disjoint sets* $A_1, A_2, \cdots, A_n$ *of integers such that for each* j, $1 \leqq j \leqq n$, $\|\Sigma_{i \in A_j} x_j(i) e_i - x_j\| \leqq \varepsilon$ *where* $x(i)$ *denote the i-th coordinate of the element* $x \in l_1$ *and* $\varepsilon = 2 - 2[1-(2+2^{\frac{1}{2}})(1-\lambda^{-1})]\lambda^{-2}$.

PROOF OF THE THEOREM. Since X is separable, there is a sequence $\{X_n\}$ of finite dimensional subspaces of X such that

$$X_n \subset X_{n+1},\ X = \overline{\bigcup_1^\infty X_n} \text{ and } d(X_n, l_1^{d(n)}) \leqq \lambda\ (d(n) = \dim X_n).$$

Hence, for each n there is a basis $x_1^n, x_2^n, \cdots, x_{d(n)}^n$ of X_n such that

$$\lambda^{-1}\left(\sum_1^{d(n)} |a_i|\right) \leqq \left\| \sum_1^{d(n)} a_i x_i^n \right\| \leqq \sum_{i=1}^{d(n)} |a_i|$$

for any sequence $\{a_i\}_1^{d(n)}$ of scalars. It is our purpose to construct a sequence $\{s(N)\}$ of integers and a sequence $\{D(N)\}$ of finite sets of integers where $D(N) \subset \{1, 2, 3, \cdots, d(s(N))\}$ such that the following two conditions are satisfied:

(*) the countable collection $\bigcup_{N=1}^\infty \{x_h^{s(N)}\}_{h \in D(N)}$ forms a basis equivalent to the unit vector basis of l_1 (that is,

$$\sum_{N=1}^\infty \sum_{h \in D(N)} |a_h^N| \geqq \left\| \sum_{N=1}^\infty \sum_{h \in D(N)} a_h^N x_h^{s(N)} \right\| \geqq \eta \sum_{N=1}^\infty \sum_{h \in D(N)} |a_n|$$

for all scalars a_h^N, where η is a constant >0) and

$$(**)\qquad X = \operatorname{span} \bigcup_{N=1}^{\infty} \{x_h^{s(N)}\}_{h\in D(N)}.$$

These goals will be achieved by a limit process under which we will construct a collection C of elements $\{u_\alpha\}$ of l_1 (not necessarily of X) which have mutually disjoint supports, have norm $\|u_\alpha\| \geqq \frac{3}{4}$. These elements will certainly satisfy the inequality

$$\sum_1^m |a_i| \geqq \left\| \sum_1^m a_i u_{\alpha_i} \right\| \geqq \frac{3}{4} \sum_1^m |a_i|$$

for every m, every distinct $\alpha_1, \alpha_2, \cdots, \alpha_m$ and any scalars $a_1, a_2, \cdots, a_m$. The process is done in such a way that for each u_α there is a basis element $x_\alpha = x_i^{s(n)}$ such that $\|u_\alpha - x_\alpha\| < \frac{1}{2}$ for all α and span $\{x_\alpha\} = X$. The above inequality certainly implies that $\{x_\alpha\}$ is a basis of X equivalent to the unit vector basis. Unfortunately, the notations we use are quite complicated because we have to pass to subsequences many times and we have to reorder the basis elements $x_1^n, x_2^n, \cdots, x_{d(n)}^n$ in some uniform way.

First note that we may apply Proposition 3 to get for each n, disjoint sets $A_1^n, A_2^n, \cdots, A_{d(n)}^n$ of integers such that

$$(1)\qquad \left\| x_j^n - \sum_{i\in A_j^n} x_j^n(i) e_i \right\| \leqq \varepsilon.$$

Note that $\varepsilon = \varepsilon(\lambda) \to 0$ as $\to 1$. We may certainly assume that A_i^n is a finite set. Next we fix n and i, $1 \leqq i \leqq d(n)$, and treat the basis element x_i^n. For $k > n$, $X_k \supset X_n$ and the basis $\{x_i^k\}_{i=1}^{d(k)}$ of X_k is "close" to the usual unit vector basis of $l_1^{d(k)}$. Therefore, after suitable changes of the constants we can use Proposition 3, with x_j replaced by x_j^n and e_i replaced by x_i^k, to get for each $k > n$ mutually disjoint subsets $B_1^{n,k}, B_2^{n,k}, \cdots, B_{d(n)}^{n,k}$ of $\{1, 2, \cdots, d(k)\}$ such that for each i, $1 \leqq i \leqq d(n)$, there are scalars $\{b_h\}_{h\in B_i^{n,k}}$ (where $b_h = b_{h,i}^{n,k}$) satisfying

$$(2)\qquad \left\| x_i^n - \sum_{h\in B_i^{n,k}} b_h x_h^k \right\| \leqq 2 - 2[1 - (2 + 2^{\frac{1}{2}})(1 - \lambda^{-2})]\lambda^{-4} = \mu,$$

where $\mu = \mu(\lambda) \to 0$ as $\lambda \to 1$.

Let

$$y_h^k = \sum_{p\in A_h^k} x_h^k(p) e_p$$

and for $k > n$ let

$$C_i^{n,k} = \{h \in B_i^{n,k} : \sum_{p\in A_i^n} |y_h^k(p)| \geqq \tfrac{3}{4}\}.$$

Also, for each $k > n$ and $h \in C_i^{n,k}$ put

$$z_h^{k,i} = \sum_{p \in A_i^n} y_h^k(p) e_p.$$

Note that $z_h^{k,i}$ depends also on n, but we will omit this additional notational complication while bearing in mind the fact that this process is done for fixed n. Clearly $\|z_h^{k,i}\| \geqq \frac{3}{4}$, and since $A_{h_1}^k \cap A_{h_2}^k = \emptyset$ if $h_1 \neq h_2$, we get that $z_{h_1}^{k,i}$ and $z_{h_2}^{k,i}$ have disjoint supports, i.e. $z_{h_1}^{k,i}(p) z_{h_2}^{k,i}(p) = 0$ for all p. Also, for each $k > n$, $1 \leqq i \leqq d(n)$ and

$$h \in C_i^{n,k},\ z_h^{k,i} \in \operatorname{span}\{e_h\}_{h \in \cup_{i=1}^{d(n)} A_i^n}$$

and since each A_i^n, $1 \leqq i \leqq d(n)$, is a finite set we can select a subsequence $\{k(m)\}_{m=1}^{\infty}$ of integers such that for each fixed i, $1 \leqq i \leqq d(n)$, the number of elements of $C_i^{n,k(m)}$ is a constant N_i^n say, for all $m \geqq 1$ and there exist elements $u_1^{n,i}, u_2^{n,i}, \cdots, u_{N_i^n}^{n,i}$ in l_1 such that

$$u_h^{n,i} = \lim_m z_{q(h)}^{k(m),i}$$

and, for $m \geqq 1$ and $1 \leqq h \leqq N_i^n$,

$$\|u_h^{n,i} - z_{q(h)}^{k(m),i}\| < \delta$$

where $\delta > 0$ is a given small positive number and $q(h)$ is the hth smallest member of $C_i^{n,k(m)}$. The above "separation" process of x_i^n will be used in the sequel for various values of n and i. We now need several estimates on the distances between the elements mentioned above. It follows from (1) and (2) that for $k > n$ we get the inequality

(3) $$\Big\| y_i^n - \sum_{h \in B_i^{n,k}} b_h y_h^k \Big\| \leqq \mu + \varepsilon + \lambda\varepsilon.$$

Hence, we have that

$$\sum_{p \notin A_i^n} \Big| y_i^n(p) - \sum_{h \in B_i^{n,k}} b_h y_h^k(p) \Big| \leqq \mu + \varepsilon + \lambda\varepsilon$$

and, therefore, the definition of $C_i^{n,k}$ ensures that

$$\Big\| \tfrac{1}{4} \sum_{h \in B_i^{n,k} - C_i^{n,k}} b_h y_h^k \Big\| \leqq \mu + \varepsilon + \lambda\varepsilon.$$

It follows that for each $k > n$

(4) $$\Big\| y_i^n - \sum_{h \in C_i^{n,k}} b_h y_h^k \Big\| \leqq 5(\mu + \varepsilon + \lambda\varepsilon)$$

and therefore, in view of (1), we have that

$$(5)\qquad \left\| x_i^n - \sum_{h\in C_i^{n,k}} b_h x_h^k \right\| \leqq 5(\mu+\varepsilon+\lambda\varepsilon)+\varepsilon+\lambda\varepsilon$$

$$= 5\mu + 6(\lambda+1)\varepsilon .$$

Also, for $m, m' \geqq 1$, if $q(h)$ and $q'(h)$ denote the h-smallest member of $C_i^{n,k(m)}$ and $C_i^{n,k(m')}$ respectively, then

$$(6)\qquad \| z_{q(h)}^{k(m),i} - z_{q'(h)}^{k(m'),i} \| \leqq \| z_{q(h)}^{k(m),i} - u_h^{n,i} \|$$

$$+ \| u_h^{n,i} - z_{q'(h)}^{k(m'),i} \| \leqq 2\delta .$$

Because $\| y_h^k - z_h^{k,i} \| \leqq \frac{1}{4}$, the inequality (6) yields the following

$$(7)\qquad \| x_{q(h)}^{k(m)} - x_{q'(h)}^{k(m')} \| \leqq \| x_{q(h)}^{k(m)} - y_{q(h)}^{k(m)} \|$$

$$+ \| y_{q(h)}^{k(m)} - z_{q(h)}^{k(m),i} \| + \| z_{q(h)}^{k(m),i} - z_{q'(h)}^{k(m'),i} \|$$

$$+ \| z_{q'(h)}^{k(m'),i} - y_{q'(h)}^{k(m')} \| + \| y_{q'(h)}^{k(m')} - x_{q'(h)}^{k(m')} \|$$

$$\leqq \tfrac{1}{2} + 2\varepsilon + 2\delta .$$

We also have that

$$(8)\qquad \| u_h^{n,i} \| \geqq \tfrac{3}{4}.$$

Let us now construct a sequence $\{s(m)\}$ of integers, a sequence $\{B(m)\}$ of finite sets of integers and a sequence $\{C(m)\}_{m=1}^{\infty}$ of collections $C(m)$ of finitely many elements of l_1 as follows. Put $s(0)=1$, $B(0)=\{1,2,\cdots,d(s(0))\}$ and use the above process with $n=1$ to select the subsequence $\{k(m)\}$, to be denoted $\{k(0,m)\}$, as above. Then put

$$C(0) = \bigcup_{i\in B(0)} \{u_h^{1,i}\}_{h=1}^{N_i^1}.$$

Let $s(1) = k(0,1)$ and put

$$B(1) = \{1,2,\cdots,d(s(1))\} \setminus \bigcup_{i\in B(0)} C_i^{s(0),s(1)},$$

repeating the above process for $n = s(1)$. Select now a subsequence $\{k(1,m)\}$ of $\{k(0,m)\}$ with $k(1,1) > s(1)$, such that for each $i \in B(1)$ the number of elements of $C_i^{s(1),k(1,m)}$ is fixed, $N_i^{s(1)}$ say,

$$u_h^{s(1),i} = \lim_m z_{q(h)}^{k(1,m),i},$$

where $q(h)$ is the h-smallest integer in $C_i^{s(1),k(1,m)}$ and

$$\|u_h^{s(1),i} - z_{q(h)}^{k(1,m),i}\| \leqq \delta$$

for all $m \geqq 1$, $i \in B(1)$ and $1 \leqq h \leqq N_i^{s(1)}$. Now put

$$C(1) = \bigcup_{i \in B(1)} \{u_h^{s(1),i}\}_{h=1}^{N_i^{s(1)}}.$$

Next we put $s(2) = k(1,1)$ and

$$B(2) = \{1, 2, \cdots, d(s(2))\} \setminus \bigcup_{j=0}^{1} \bigcup_{i \in B(j)} C_i^{s(j),s(2)}$$

and use the above process for $n = s(2)$ to select the collection $C(2)$. In general, $s(N+1) = k(N,1)$,

$$B(N+1) = \{1, 2, \cdots d(s(N+1))\} - \bigcup_{j=1}^{N} \bigcup_{i \in B(j)} C_i^{s(j),s(N+1)}$$

and

$$C(N+1) = \bigcup_{i \in B(N+1)} \{u_h^{s(N+1),i}\}_{h=1}^{N_i^{s(N+1)}},$$

where we have that $k(N+1,1) > s(N+1)$ and

(9)
$$\|u_h^{s(N+1),i} - z_{q(h)}^{k(N+1,m),i}\| \leqq \delta \quad \text{for all}$$
$$m \geqq 1, \quad i \in B(N+1) \quad \text{and} \quad 1 \leqq h \leqq N_i^{s(N+1)}$$

We are now ready to show that X is isomorphic to l_1. First note that $C = \bigcup_1^\infty C(N)$ is a collection of elements u of l_1 with norm $\|u\| \geqq \frac{3}{4}$ and the construction ensures that any $u_1, u_2 \in C$ have disjoint supports, i.e. $u_1(p)u_2(p) = 0$ for all p. It follows from (8) that for any distinct $u_1, u_2, \cdots, u_n \in C$ and any scalars $a_1, a_2, \cdots, a_n$ we have that

(10)
$$\sum_1^n |a_i| \geqq \left\| \sum_1^n a_i u_i \right\| \geqq \tfrac{3}{4} \sum_1^n |a_i|.$$

However, for each $N, i \in B(N-1)$ and $1 \leqq h \leqq N_i^{s(N-1)}$ ($q(h) \in C_i^{s(N-1),s(N)}$), we have by (9) and (1) that

(11)
$$\begin{aligned}\|u_h^{s(N),i} - x_{q(h)}^{s(N)}\| &\leqq \|u_h^{s(N),i} - z_{q(h)}^{s(N),i}\| \\ &\quad + \|z_{q(h)}^{s(N),i} - y_{q(h)}^{s(N)}\| \\ &\quad + \|y_{q(h)}^{s(N)} - x_{q(h)}^{s(N)}\| \\ &\leqq \delta + \tfrac{1}{4} + \varepsilon.\end{aligned}$$

We can now put

$$D(N) = \bigcup_{i \in B(N-1)} C_i^{s(N-1),s(N)},$$

then show that conditions (*) and (**) are satisfied. Indeed, it follows from (10) and (11) that for any n and any collection of scalars

$$\bigcup_{N=1}^{n} \bigcup_{i \in B(N-1)} \{a_h^N\}_{h \in C_i^{s(N-1),s(N)}},$$

we have that

$$\sum_{N=1}^{n} \sum_{i \in B(N-1)} \sum_{h \in C_i^{s(N-1),s(N)}} |a_h^N| \tag{12}$$

$$\geqq \left\| \sum_{N=1}^{n} \sum_{i \in B(N-1)} \sum_{h \in C_i^{s(N-1),s(N)}} a_h^N x_h^{s(N)} \right\|$$

$$\geqq \left(\sum_{N=1}^{n} \sum_{i \in B(N-1)} \sum_{h \in C_i^{s(N-1),s(N)}} |a_h^N| \right) (\tfrac{1}{2} - \delta - \varepsilon),$$

hence the collection

$$K = \bigcup_{N=1}^{\infty} \bigcup_{i \in B(N-1)} \{x_h^{s(N)}\}_{h \in C_i^{s(N-1),s(N)}}$$

forms in $Y =$ span K a basis equivalent to the unit vector basis of l_1. Y is a subspace of X and we would like to show that $Y = X$. To do so it is certainly enough to show that for each $x \in \bigcup_{N=1}^{\infty} X_{s(N)}$ with $\|x\| = 1$, $\inf_{y \in Y} \|x - y\| \leqq M < 1$, where M is independent on x. Let

$$x = \sum_{i=1}^{d(s(N))} a_i x_i^{s(N)}$$

and assume that $\|x\| = 1$ (and hence $\Sigma_1^{d(s(N))} |a_i| \leqq \lambda$). Obviously

$$\{1, 2, \cdots, d(s(N))\} \setminus B(N) = \bigcup_{j=1}^{N-1} \bigcup_{i \in B(j)} C_i^{s(j),s(N)},$$

where all the sets appearing in the union are mutually disjoint. Moreover, for each $1 \leqq j \leqq N-1$, $i \in B(j)$ and $1 \leqq h \leqq N_i^{s(j+1)} = N_i^{s(N)}$, if $q(h)$ and $q'(h)$ denote the h-smallest members of $C_i^{s(j),s(j+1)}$ and $C_i^{s(j),s(N)}$ respectively, then, by (7), we get that

$$\|x_{q(h)}^{s(j+1)} - x_{q'(h)}^{s(N)}\| \leqq \tfrac{1}{2} + 2\varepsilon + 2\delta. \tag{13}$$

On the other hand, for each $i \in B(N)$ we have, by (5), that

$$\left\| x_i^{s(N)} - \sum_{h \in C_i^{s(N),s(N+1)}} b_h x_h^{s(N+1)} \right\| \leqq 5\mu + 6(\lambda + 1)\varepsilon. \tag{14}$$

Put $M = \mathrm{Max}\{\lambda(\frac{1}{2} + 2\varepsilon + 2\delta), \lambda(5\mu + 6(\lambda + 1)\varepsilon)\}$, then we have by (13) and (14) that

$$\left\| x - \sum_{j=1}^{N} \sum_{i \in B(j-1)} \sum_{h \in C_i^{s(j-1),s(j)}} a_h x_h^{s(j)} - \sum_{i \in B(N)} a_i \left(\sum_{h \in C_i^{s(N),s(N+1)}} b_h x_h^{s(N+1)} \right) \right\| \leqq M < 1.$$

This completes the proof of the theorem.

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